Quantization of free fermions

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$$\mathcal{L}_D = \bar{\Psi}(i\gamma^\mu \partial_\mu - m)\Psi \qquad \gamma^\mu\partial_\mu \equiv \not{\partial}$$

$$\bar{\Psi} = \Psi^+ \gamma_0 \qquad \{\gamma_\mu, \gamma_\nu\} = 2\eta_{\mu\nu}\ , \qquad \gamma_5 \equiv i\gamma^0\gamma^1\gamma^2\gamma^3$$

$$(\gamma^0)^+ = \gamma^0 \qquad \gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \tilde{\sigma}^\mu & 0 \end{pmatrix} \qquad \sigma^\mu \equiv (\mathbb{1}, \sigma^i) \qquad \sigma^i \text{ - Pauli matrices}$$

$$\gamma^{i+} = -\gamma^i \qquad \tilde{\sigma}^\mu \equiv (\mathbb{1}, -\sigma^i) \qquad \sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

- Show that in the standard (chiral) representation

$$\sigma^i\sigma^j = \delta^{ij} + i\epsilon^{ijk}\sigma^k$$

$$\gamma_5 = \begin{pmatrix} -\mathbb{1} & 0 \\ 0 & \mathbb{1} \end{pmatrix}$$

and that $\gamma^{5\,2} = \mathbb{1}$, $(\gamma 5)^+ = \gamma^5$, $\{\gamma^5, \gamma^\mu\} = 0$

$$\gamma^0\gamma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & \sigma^1 \\ -\sigma^1 & 0 \end{pmatrix} = \begin{pmatrix} -\sigma^1 & 0 \\ 0 & \sigma^1 \end{pmatrix}, \quad \gamma^2\gamma^3 = \begin{pmatrix} 0 & \sigma^2 \\ -\sigma^2 & 0 \end{pmatrix}\begin{pmatrix} 0 & \sigma^3 \\ -\sigma^3 & 0 \end{pmatrix} = \begin{pmatrix} -\sigma^2\sigma^3 & 0 \\ 0 & -\sigma^2\sigma^3 \end{pmatrix}$$

$$P_{L/R} = \frac{1}{2}(1 \mp \gamma_5) \qquad P_L = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & 0 \end{pmatrix} \qquad P_R = \begin{pmatrix} 0 & 0 \\ 0 & \mathbb{1} \end{pmatrix} \quad \text{- chiral projectors}$$

$$P_L + P_R = \mathbb{1}\ , \qquad P_L P_R = P_R P_L = 0$$

$$P_{L/R}^2 = P_{L/R}$$

$$\Psi = \begin{pmatrix} \varphi \\ \lambda \end{pmatrix} \qquad P_L\Psi = \begin{pmatrix} \varphi \\ 0 \end{pmatrix} \qquad P_R\Psi = \begin{pmatrix} 0 \\ \lambda \end{pmatrix}$$

$$\Psi_L \equiv \begin{pmatrix} \varphi \\ 0 \end{pmatrix} \qquad \Psi_R \equiv \begin{pmatrix} 0 \\ \lambda \end{pmatrix} \qquad \Rightarrow \qquad P_{L/R}\Psi_{L/R} = \Psi_{L/R}$$

- Decompose the current $j^\mu = \bar\psi \gamma^\mu \psi$ into chiral components

$$\bar\psi \gamma^\mu \psi = (\overline{\psi_L + \psi_R})\,\gamma^\mu (\psi_L + \psi_R) = \bar\psi_L \gamma^\mu \psi + \bar\psi_R \gamma^\mu \psi_R$$

$$\bar\psi_L \gamma^\mu \psi_R = (\overline{P_L \psi})\,\gamma^\mu P_R \psi = \psi^\dagger P_L^\dagger \gamma_0 \gamma^\mu P_R \psi = \bar\psi \underbrace{\gamma_0 P_L \gamma_0}_{P_R} \gamma^\mu P_R \psi = \bar\psi \underbrace{P_R \gamma^\mu P_R}_{\gamma^\mu P_L P_R = 0} \psi = 0$$

- Decompose the mass term into chiral components

$$\bar\psi \psi = (\overline{\psi_L + \psi_R})(\psi_L + \psi_R) = \bar\psi_L \psi_R + \bar\psi_R \psi_L$$

$$\bar\psi_L \psi_L = (\overline{P_L \psi})\, P_L \psi = \psi^\dagger P_L^\dagger \gamma_0 P_L \psi = \bar\psi P_R P_L \psi = 0$$

Solutions of the Dirac equations

$(i\not\partial - m)\psi = 0$, plane wave solutions $\psi(x) = \begin{cases} u(p)\, e^{-ipx} & \text{positive energy} \\ v(p)\, e^{ipx} & \text{negative energy} \end{cases}$

$$\Downarrow$$

$$\begin{aligned} (\not p - m)\, u(p) &= 0 \\ (\not p + m)\, v(p) &= 0 \end{aligned} \quad \Leftarrow \quad \begin{aligned} [i\gamma^\mu(-ip_\mu) - m]\, u(p)\, e^{-ipx} &= 0 \\ [i\gamma^\mu(ip_\mu) - m]\, v(p)\, e^{ipx} &= 0 \end{aligned}$$

$u(p) = \begin{pmatrix} u_L(p) \\ u_R(p) \end{pmatrix}$ assume $m \neq 0$, in the rest frame $\bar p = (m, 0, 0, 0)$

$$\Downarrow$$

$$\boxed{(\gamma^0 - \mathbb{1})\, u(p) = 0}$$

$$(\gamma^0 + \mathbb{1})\, v(p) = 0$$

$$\gamma^0 - \mathbb{1} = \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix} - \begin{pmatrix} \mathbb{1} & 0 \\ 0 & \mathbb{1} \end{pmatrix} = \begin{pmatrix} -\mathbb{1} & \mathbb{1} \\ \mathbb{1} & -\mathbb{1} \end{pmatrix}$$

$$\begin{pmatrix} -1 & 1 \\ \end{pmatrix} \begin{pmatrix} u_L \\ \end{pmatrix} = 0 \Rightarrow u_L = u_R$$

$$(\gamma^0 + 1)\, v(p) = 0$$

$$\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} u_L \\ u_R \end{pmatrix} = 0 \quad \Rightarrow \quad u_L = u_R$$

we can choose the normalization $u_L = u_R = \sqrt{m}\,\xi$, $\xi^\dagger \xi = 1$

For generic $p = (E, 0, 0, p)$ one gets

$$u^s(p) = \begin{pmatrix} \left[(E+p)^{1/2}\frac{1}{2}(1-\sigma^3) + (E-p)^{1/2}\frac{1}{2}(1+\sigma^3)\right]\xi^s \\ \left[(E+p)^{1/2}\frac{1}{2}(1+\sigma^3) + (E-p)^{1/2}\frac{1}{2}(1-\sigma^3)\right]\xi^s \end{pmatrix} \quad \text{for } \xi^1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \; \xi^2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

in the ultrarelativistic limit $p^\mu \to (E, 0, 0, E)$, then

$$u^1(p) \to (2E)^{1/2}\begin{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} \end{pmatrix} = (2E)^{1/2}\begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} = (2E)^{1/2}\begin{pmatrix} 0 \\ \xi^1 \end{pmatrix} \sim \text{righthanded}$$

$$u^2(p) \to \qquad = (2E)^{1/2}\begin{pmatrix} \xi^2 \\ 0 \end{pmatrix} \sim \text{lefthanded}$$

$$v^s(p) = \begin{pmatrix} \left[(E+p)^{1/2}\frac{1}{2}(1-\sigma^3) + (E-p)^{1/2}\frac{1}{2}(1+\sigma^3)\right]\eta^s \\ -\left[(E+p)^{1/2}\frac{1}{2}(1+\sigma^3) + (E-p)^{1/2}\frac{1}{2}(1-\sigma^3)\right]\eta^s \end{pmatrix} \quad \text{for } \eta^1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \; \eta^2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Symmetries of $\mathcal{L}_D$

$$\mathcal{L}_D = \bar\psi(i\partial\!\!\!/ - m)\psi = \bar\psi_L i\partial\!\!\!/ \psi_L + \bar\psi_R i\partial\!\!\!/ \psi_R - m(\bar\psi_L\psi_R + \bar\psi_R\psi_L)$$

$$\psi_i \to e^{i\theta_i}\psi_i \qquad i = L, R$$

1) if $m = 0$ then $U(1) \times U(1)$

$\psi_i \to e \quad \psi_i \qquad i = L, R$

1) if $m = 0$ then $U(1) \times U(1)$

2) if $m \neq 0$ then $U(1)$ for $\theta_L = \theta_R$

$$\left.\begin{array}{l} \psi_L \to e^{i\theta_L} \psi_L \\ \psi_R \to e^{i\theta_R} \psi_R \end{array}\right| \Rightarrow \psi \, (= \psi_L + \psi_R) \to e^{i\theta_L} \psi_L + e^{i\theta_R} \psi_R = \left[\frac{1}{2}(1-\gamma_5) e^{i\theta_L} + \frac{1}{2}(1+\gamma_5) e^{i\theta_R}\right] \psi =$$

$$e^{i\beta\gamma^5} = \sum_{n=0}^{\infty} \frac{(i\beta\gamma^5)^n}{n!} = \sum_{n=0}^{\infty} \frac{(i\beta\gamma^5)^{2n}}{(2n)!} + \sum_{n=0}^{\infty} \frac{(i\beta\gamma^5)^{2n+1}}{(2n+1)!} = \mathbb{1} \underbrace{\sum_{n=0} \frac{(-1)^n \beta^{2n}}{(2n)!}}_{\cos\beta} + \gamma^5 i \underbrace{\sum_{n=0} \frac{(-1)^n \beta^{2n+1}}{(2n+1)!}}_{\sin\beta} =$$

$$= \mathbb{1} \cos\beta + i\gamma^5 \sin\beta$$

$$= \frac{1}{2}\left[e^{i\theta_L} + e^{i\theta_R} - \gamma^5\left(e^{i\theta_L} - e^{i\theta_R}\right)\right]\psi = \frac{1}{2} e^{i\alpha}\left[e^{i\beta} + e^{-i\beta} - \gamma^5\left(e^{i\beta} - e^{-i\beta}\right)\right]\psi =$$

$$\alpha = \frac{1}{2}(\theta_L + \theta_R) \qquad \theta_L = \alpha + \beta \qquad = e^{i\alpha}\left(\cos\beta - i\gamma^5 \sin\beta\right)\psi = e^{i\alpha} e^{-i\beta\gamma_5} \psi =$$

$$\beta = \frac{1}{2}(\theta_L - \theta_R) \qquad \theta_R = \alpha - \beta \qquad = e^{i(\alpha - \beta\gamma_5)} \psi$$

1) $\psi \to e^{i\alpha} \psi$ (always symmetry) (vector $U(1)$)

2) $\psi \to e^{-i\gamma_5\beta} \psi$ (symmetry for $m=0$) chiral transformation (axial $U(1)$)

$\Downarrow$

Noether currents $j^\mu_a = -\dfrac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_i)} F_{i,a}$ for $\phi_i \longrightarrow \phi_i + \epsilon^a F_{i,a}(\phi, \partial\phi)$

$$\partial_\alpha \qquad \frac{\quad}{\partial(\partial_\mu \phi_i)}$$

$$\psi \to e^{i\alpha}\psi \Rightarrow \quad \phi_i \to \psi_\alpha, \quad \epsilon^a = \alpha, \quad F_{i,a} = i\psi_\alpha$$

$$\frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi_\alpha)} = \frac{\partial}{\partial(\partial_\mu \psi_\alpha)} \bar\psi i\gamma^\nu \partial_\nu \psi = (\bar\psi i \gamma^\mu)_\alpha$$

$$j_V^\mu = \bar\psi \gamma^\mu \psi \quad , \quad \partial_\mu j_V^\mu = 0$$

$$\psi \to e^{i\beta\gamma^5}\psi \Rightarrow \quad \phi_i \to \psi_\alpha, \quad \epsilon^a = \beta, \quad F_{i,a} = i\gamma_5\psi$$

$$j_A^\mu = \bar\psi \gamma^\mu \gamma_5 \psi \quad , \quad \partial_\mu j_A^\mu = 2im\bar\psi\gamma_5\psi$$

– Show that $\partial_\mu j_A^\mu = 2im\bar\psi\gamma_5\psi$

$$\partial_\mu(\bar\psi\gamma^\mu\gamma_5\psi) = \underbrace{\overline{(\partial_\mu\psi)}\gamma^\mu\gamma_5}_{im\bar\psi\gamma_5}\psi + \bar\psi\underbrace{\gamma^\mu\gamma_5\partial_\mu\psi}_{-\gamma_5\not\partial\psi = -\gamma_5(-i)m\psi} = 2im\bar\psi\gamma_5\psi$$

$$(i\not\partial - m)\psi = 0$$

$$(i\gamma^\mu\partial_\mu - m)\psi = 0 \;|^\dagger$$

$$-i(\partial_\mu\psi)^\dagger_{\gamma_0^2}\gamma^{\mu\dagger} - m\psi^\dagger = 0 \;|\gamma_0$$

$$-i\overline{(\partial_\mu\psi)}\underbrace{\gamma_0\gamma^{\mu\dagger}\gamma_0}_{\gamma^\mu} - m\bar\psi = 0$$

$$-i\overline{\partial_\mu\psi}\gamma^\mu - m\bar\psi = 0$$

$$\gamma_0\gamma^{\mu\dagger}\gamma_0 = \begin{cases}\gamma^0 & \text{for } \mu = 0 \\ \gamma^i & \text{for } \mu = i\end{cases}$$

Quantization of free fermion

$$\mathcal{L} = \bar{\psi}(i\partial\!\!\!/ - m)\psi \quad \Rightarrow \quad (\pi_\psi)_a = \frac{\partial \mathcal{L}}{\partial(\partial_0 \psi_a)} = i(\bar{\psi}\gamma^0)_a = i(\psi^+)_a$$

Spin-statistics theorem $\Rightarrow \{\psi_a(t,\bar{x}), \psi_b^+(t,\bar{y})\} = \delta_{ab}\,\delta^{(3)}(\bar{x}-\bar{y})$, $\{\psi_a, \psi_b\} = \{\psi_a^+, \psi_b^+\} = 0$

$$\psi(x) = \int \frac{d^3p}{(2\pi)^3 (2E_p)^{1/2}} \sum_{s=1,2} \left(a_{p,s} u^s e^{-ipx} + b^+_{p,s} v^s e^{ipx} \right)$$

$u^s = u^s(p)$
$v^s = v^s(p)$

$$\bar{\psi}(x) = \int \frac{d^3p}{(2\pi)^3 (2E_p)^{1/2}} \sum_{s=1,2} \left(a^+_{p,s} \overline{u^s} e^{ipx} + b_p \overline{v^s} e^{-ipx} \right)$$

$$\underline{\{\psi_a(t,\bar{x}), \psi_b^+(t,\bar{y})\} = \delta_{ab}\,\delta^{(3)}(\bar{x}-\bar{y}) \quad , \quad \{\psi_a, \psi_b\} = \{\psi_a^+, \psi_b^+\} = 0}$$

$$\Updownarrow$$

$$\{a_p^r, a_q^{s+}\} = \{b_p^r, b_q^{s+}\} = (2\pi)^3 \delta^{(3)}(\bar{p}-\bar{q})\,\delta^{rs} \quad , \quad \{a,a\} = \{a^+,a^+\} = \{b,b\} = \{b^+,b^+\} = 0$$

vacuum : $a_{p,s}|0\rangle = b_{p,s}|0\rangle = 0$

1-particle state $(2E_p)^{1/2} a^+_{p,s}|0\rangle$, $(2E_p)^{1/2} b^+_{p,s}|0\rangle$

2-particle state $\propto a^+_{q,r} a^+_{p,s}|0\rangle = -a^+_{p,s} a^+_{q,r}|0\rangle$

2-particle state $\propto a^{\dagger}_{q,r}\, a^{\dagger}_{p,s}|0\rangle = -a^{\dagger}_{p,s}\, a^{\dagger}_{q,r}|0\rangle$

$$\{a^{\dagger}_{p,s}, a^{\dagger}_{q,r}\} = 0, \ldots \Rightarrow (a^{\dagger}_{p,s})^2 = (a_{p,s})^2 = (b^{\dagger}_{p,s})^2 = (b_{p,s})^2 = 0$$

The number operator $N_{p,r} = V^{-1} a^{\dagger}_{p,r} a_{p,r}$ $\qquad V \leftarrow (2\pi)^3 \delta^{(3)}(0)$

$$(N_{p,r})^2 = V^{-2} a^{\dagger}_{p,r} a_{p,r} a^{\dagger}_{p,r} a_{p,r} = V^{-2} a^{\dagger}_{p,r}\left(V - a^{\dagger}_{p,r} a_{p,r}\right) a_{p,r} =$$

$$= V^{-1} a^{\dagger}_{p,r} a_{p,r} = N_{p,r}$$

$$\Downarrow$$

$$N_{p,r}(N_{p,r} - 1) = 0 \Rightarrow N_{p,r} = 0, 1$$

Hamiltonian

"Classical" field theory: $\mathcal{H} = -\mathcal{L} + \frac{\partial \mathcal{L}}{\partial(\partial_0\psi)_\alpha}\partial^0\psi_\alpha = -\bar{\psi}(i\not{\partial} - m)\psi + \bar{\psi} i\gamma^0\partial_0\psi =$

$$= -\cancel{\bar{\psi} i\gamma^0\partial_0\psi} - \bar{\psi} i\gamma^i\partial_i\psi + m\bar{\psi}\psi + \cancel{\bar{\psi} i\gamma^0\partial_0\psi} =$$

$$= \bar{\psi}(-i\gamma^i\partial_i + m)\psi$$

Hamilton operator $H = \int d^3x :\bar{\psi}(-i\vec{\gamma}\cdot\vec{\nabla} + m)\psi:$, where

$:a a^{\dagger}: = -a^{\dagger}a$ and $:b b^{\dagger}: = -b^{\dagger} b$

$$\psi(x) = \int \frac{d^3p}{(2\pi)^3(2E_p)^{1/2}} \sum_s \left(a_{p,s} u^s e^{-ipx} + b^{\dagger}_{p,s} v^s e^{ipx}\right)$$

$$\Psi(x) = \int \frac{d^3p}{(2\pi)^3 (2E_p)^{1/2}} \sum_{s=1,2} \left(a_{p,s} u^s e^{-ipx} + b^+_{p,s} v^s e^{ipx} \right)$$

$$\bar\Psi(x) = \int \frac{d^3p}{(2\pi)^3 (2E_p)^{1/2}} \sum_{s=1,2} \left(a^+_{p,s} \bar{u}^s e^{ipx} + b_p \bar{v}^s e^{-ipx} \right)$$

$$H = \int d^3x \int \frac{d^3p}{(2\pi)^3 (2E_p)^{1/2}} \int \frac{d^3p'}{(2\pi)^3 (2E_{p'})^{1/2}} \sum_{s,s'=1,2} : \left(a^+_{p,s} \bar{u}^s(p) e^{ipx} + b_{p,s} \bar{v}^s(p) e^{-ipx} \right) \underbrace{\left(-i\bar\gamma\cdot\bar\nabla + m \right) \left(a_{p',s'} u^{s'}(p') e^{-ip'x} + b^+_{p',s'} v^{s'}(p') e^{ip'x} \right)} :$$

$$-i\bar\gamma(+i\bar p') \left(a_{p',s'} u^{s'}(p') e^{-ip'x} - b^+_{p',s'} v^{s}(p') e^{ip'x} \right) +$$

$$+ \quad m \quad (\qquad + \qquad) =$$

$$= (\bar\gamma \bar p' + m)\, a_{p',s'} u^{s'}(p') e^{-ip'x} - (\bar\gamma\cdot\bar p' - m)\, b^+_{p',s'} v^{s'}(p') e^{ip'x}$$

$$= \int d^3x \ \text{———} \ \text{〃} \ \text{————} \ \text{〃} \ \text{——} : \quad a^+_{p,s} a_{p',s'} \bar{u}^s(p) (\bar\gamma\cdot\bar p' + m) u^{s'}(p')\, e^{i(E_p - E_{p'})t}\, e^{-i(\bar p - \bar p')\bar x} +$$

$$- a^+_{p,s} b^+_{p',s'} \bar{u}^s(p) (\bar\gamma\cdot\bar p' - m) v^{s'}(p')\, e^{i(E_p + E_{p'})t}\, e^{-i(\bar p + \bar p')x} +$$

$$+ b_{p,s} a_{p',s'} \bar{v}^s(p) (\bar\gamma\cdot\bar p' + m) u^{s'}(p')\, e^{-i(E_p + E_{p'})t}\, e^{+i(\bar p + \bar p')x} +$$

$$- b_{p,s} b^+_{p',s'} \bar{v}^s(p) (\bar\gamma\cdot\bar p' - m) v^{s'}(p')\, e^{-i(E_p - E_{p'})t}\, e^{i(\bar p - \bar p')x} : =$$

$$= \int d^3p \sum \quad a^+ \ a \quad \bar{u}^s(p) (\bar\gamma\cdot\bar p + m) u^{s'}(p) - b \ b^+ \quad \bar{v}^s(p) (\bar\gamma\cdot\bar p - m) v^{s'}(p) \quad +$$

$$= \int \frac{d^3p}{(2\pi)^3 (2E_p)} \sum_{s,s'=1,2}' : a^+_{p,s} a_{p,s'} \bar{u}^s(p)(\bar{\gamma}\cdot\bar{p}+m)u^{s'}(p) - b_{p,s} b^+_{p,s'} \bar{v}^s(p)(\bar{\gamma}\cdot\bar{p}-m)v^{s'}(p) +$$

$$- a^+_{p,s} b^+_{-p,s'} \bar{u}^s(p)(-\bar{\gamma}\cdot\bar{p}-m)v^{s'}(-p)\, e^{2iE_pt} + b_{p,s} a_{-p,s'} \bar{v}^s(p)(-\bar{\gamma}\cdot\bar{p}+m)u^{s'}(-p)\, e^{-2iE_pt} :$$

adopting the following identities :

$$p \equiv (E_p, -\bar{p})$$

$$\bar{u}^s(p)\, u^{s'}(p) = 2m\,\delta^{ss'} \qquad \bar{v}^s(p)\, v^{s'}(p) = -2m\,\delta^{ss'}$$

$$u^{s+}(p)\, u^{s'}(p) = 2E_p\,\delta^{ss'} \qquad v^{s+}(p)\, v^{s'}(p) = 2E_p\,\delta^{ss'}$$

$$\bar{u}^s(p)\, v^{s'}(p) = 0 \qquad \bar{v}^s(p)\, u^{s'}(p) = 0$$

and

$$(\not{p} - m)\, u^s(p) = 0 \qquad \bar{u}^s(p)(\not{p} - m) = 0$$

$$(\not{p} + m)\, v^s(p) = 0 \qquad \bar{v}^s(p)(\not{p} + m) = 0$$

we find

$$(\gamma^0 E_p - \bar{\gamma}\cdot\bar{p} - m)\, u^s(p) = 0 \quad \rightarrow \quad (\bar{\gamma}\cdot\bar{p}+m)\, u^{s'}(p) = \gamma^0 E_p\, u^{s'}(p)$$

$$(-\bar{\gamma}\cdot\bar{p}+m)\, u^{s'}(-p) = \gamma^0 E_p\, u^{s'}(-p)$$

$$(\gamma^0 E_p - \bar{\gamma}\cdot\bar{p} + m)\, v^s(p) = 0 \quad \rightarrow \quad (\bar{\gamma}\cdot\bar{p}-m)\, v^{s'}(p) = \gamma^0 E_p\, v^{s'}(p)$$

$$(-\bar{\gamma}\cdot\bar{p}-m)\, v^{s'}(-p) = \gamma^0 E_p\, v^{s'}(-p)$$

$$H = \int \frac{d^3p}{(2\pi)^3 (2E_p)} \sum_{s,s'=1,2}' : a^+_{p,s} a_{p,s'} \bar{u}^s(p)\underbrace{(\bar{\gamma}\cdot\bar{p}+m)u^{s'}(p)}_{\gamma^0 E_p u^{s'}(p)} - b_{p,s} b^+_{p,s'} \bar{v}^s(p)\underbrace{(\bar{\gamma}\cdot\bar{p}-m)v^{s'}(p)}_{\gamma^0 E_p v^{s'}(p)} +$$

$$- a^+_{p,s} b^+_{-p,s'} \bar{u}^s(p)\underbrace{(-\bar{\gamma}\cdot\bar{p}-m)v^{s'}(-p)}_{\gamma^0 E_p v^{s'}(-p)}\, e^{2iE_pt} + b_{p,s} a_{-p,s'} \bar{v}^s(p)\underbrace{(-\bar{\gamma}\cdot\bar{p}+m)u^{s'}(-p)}_{\gamma^0 E_p u^{s'}(-p)}\, e^{-2iE_pt} : =$$

$$\qquad\qquad _{p,s} \quad _{-p,s'} \quad \underbrace{\qquad\qquad}_{\gamma^0 E_p v^{s'}(-p)} \qquad\qquad _{p,s} \quad _{-p,s'} \quad \underbrace{\qquad\qquad}_{\gamma^0 E_p u^{s'}(-p)}$$

$$= \int \frac{d^3p}{(2\pi)^3 (2E_p)} E_p \sum_{s,s'} : a^+_{p,s} a_{p,s'} \underbrace{u^{s+}(p)\, u^{s'}(p)}_{2E_p \delta^{ss'}} - b_{p,s} b^+_{p,s'} \underbrace{v^{s+}(p)\, v^{s'}(p)}_{2E_p \delta^{ss'}} - a^+_{p,s} b^+_{-p,s'} \underbrace{u^{s+}(p)\, v^{s'}(-p)}_{=0} e^{2iE_p t} + b_{p,s} a_{-p,s'} \underbrace{v^{s+}(p)\, u^{s'}(-p)}_{=0} e^{-2iE_p t} :$$

$$u^s(p) = \begin{pmatrix} \left[(E+p)^{1/2} \frac{1}{2}(1-\sigma^3) + (E-p)^{1/2} \frac{1}{2}(1+\sigma^3)\right] \xi^s \\ \left[(E+p)^{1/2} \frac{1}{2}(1+\sigma^3) + (E-p)^{1/2} \frac{1}{2}(1-\sigma^3)\right] \xi^s \end{pmatrix}$$

$$u^s(p)^+ = \left(\xi^{s+}\left[(E+p)^{1/2} \frac{1}{2}(1-\sigma^3) + (E-p)^{1/2} \frac{1}{2}(1+\sigma^3)\right], \xi^{s+}\left[(E+p)^{1/2} \frac{1}{2}(1+\sigma^3) + (E-p)^{1/2} \frac{1}{2}(1-\sigma^3)\right] \right)$$

$$v^s(-p) = \begin{pmatrix} \left[(E-p)^{1/2} \frac{1}{2}(1-\sigma^3) + (E+p)^{1/2} \frac{1}{2}(1+\sigma^3)\right] \eta^s \\ -\left[(E-p)^{1/2} \frac{1}{2}(1+\sigma^3) + (E+p)^{1/2} \frac{1}{2}(1-\sigma^3)\right] \eta^s \end{pmatrix} \Leftarrow v^s(p) = \begin{pmatrix} \left[(E+p)^{1/2} \frac{1}{2}(1-\sigma^3) + (E-p)^{1/2} \frac{1}{2}(1+\sigma^3)\right] \eta^s \\ -\left[(E+p)^{1/2} \frac{1}{2}(1+\sigma^3) + (E-p)^{1/2} \frac{1}{2}(1-\sigma^3)\right] \eta^s \end{pmatrix}$$

$$u^s(p)^+ \cdot v^{s'}(-p) = \xi^{s+}\left[(E+p)^{1/2} \frac{1}{2}(1-\sigma^3) + (E-p)^{1/2} \frac{1}{2}(1+\sigma^3)\right]\left[(E-p)^{1/2} \frac{1}{2}(1-\sigma^3) + (E+p)^{1/2} \frac{1}{2}(1+\sigma^3)\right] \eta^{s'} +$$

$$- \xi^{s+}\left[(E+p)^{1/2} \frac{1}{2}(1+\sigma^3) + (E-p)^{1/2} \frac{1}{2}(1-\sigma^3)\right]\left[(E-p)^{1/2} \frac{1}{2}(1+\sigma^3) + (E+p)^{1/2} \frac{1}{2}(1-\sigma^3)\right] \eta^{s'} =$$

$$(1-\sigma^3)(1+\sigma^3) = 0$$

$$(1\mp\sigma^3)^2 = 1 \mp 2\sigma^3 + \sigma^{3\,2} = 2(1\pm\sigma^3)$$

$$= \xi^{s\dagger}\Big[\underbrace{\sqrt{(E+p)(E-p)}}_{m}\Big\{\frac{(1-\cancel{\sigma^3})}{2} + \frac{(1+\cancel{\sigma^3})}{2}\Big\}\Big]\eta^{s'} - \xi^{s\dagger}\Big\{\underbrace{\sqrt{(E-p)(E+p)}}_{m}\Big[\frac{1+\cancel{\sigma^3}}{2} + \frac{1-\cancel{\sigma^3}}{2}\Big]\eta^{s'} = 0$$

$$\boxed{H = \int \frac{d^3p}{(2\pi)^3} E_p \sum_s \left(a^\dagger_{p,s}\, a_{p,s} + b^\dagger_{p,s}\, b_{p,s}\right)}$$

- what would has happened if we had used commutators instead of anticommutators?

$$H = \int \frac{d^3p}{(2\pi)^3} E_p \sum_s \left(a^\dagger_{p,s}\, a_{p,s} - b^\dagger_{p,s}\, b_{p,s}\right)$$

$$-b^\dagger_{p,s}\, b_{p,s}\, b^\dagger_{q,t}|0\rangle = -b^\dagger_{p,s}\Big(\underbrace{[b_{p,s}, b^\dagger_{q,t}]}_{(2\pi)^3\delta^{(3)}(\bar p-\bar q)\,\delta_{s,t}} + b^\dagger_{q,t}\, b_{p,s}\Big)|0\rangle$$

$$-\int \frac{d^3p}{(2\pi)^3} E_p \sum_s b^\dagger_{p,s}\, b_{p,s}\, b^\dagger_{q,t}|0\rangle = -\int d^3p\, E_p\, \delta^{(3)}(\bar p-\bar q)\sum_s \delta_{s,t}\, b^\dagger_{p,s}|0\rangle = -E_q\, b^\dagger_{q,t}|0\rangle$$

$$\Downarrow$$

addition of an extra particle created by $b^\dagger_{q,t}$ decreases energy by $-E_q$, so H would be unbounded from below

$$\bar{P} = \int \frac{d^3p}{(2\pi)^3} \sum_{s=1,2} \bar{p} \left(a^+_{p,s} a_{p,s} + b^+_{p,s} b_{p,s} \right)$$

$$Q_{U(1)} = \int \frac{d^3p}{(2\pi)^3} \sum_{s=1,2} \left(a^+_{p,s} a_{p,s} - b^+_{p,s} b_{p,s} \right) \quad \Leftarrow \quad \psi \to e^{i\alpha} \psi$$